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Constraints on Unified Gauge Theories from Noncommutative Geometry

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Abstract

The Connes and Lott reformulation of the strong and electroweak model represents a promising application of noncommutative geometry. In this scheme the Higgs field naturally appears in the theory as a particular *gauge boson*, connected to the discrete internal space, and its quartic potential, fixed by the model, is not vanishing only when more than one fermion generation is present. Moreover, the exact hypercharge assignments and relations among the masses of particles have been obtained. This paper analyzes the possibility of extensions of this model to larger unified gauge groups. Noncommutative geometry imposes very stringent constraints on the possible theories, and remarkably, the analysis seems to suggest that no larger gauge groups are compatible with the noncommutative structure, unless one enlarges the fermionic degrees of freedom, namely the number of particles.

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1 Introduction

There seems to be little doubt that Yang–Mills theories provide the correct framework to describe physical interactions at the elementary particle level. Nevertheless the realistic model of the fundamental interactions (gravity excluded) which has been built according to these ideas, the so-called standard model (SM), still contains features which are not completely satisfactory. This would suggest the presence of deeper unifications at scales higher than the electroweak one, based on larger gauge groups. Actually, there are cosmological open problems, i.e. the baryon asymmetry of the universe, the inflationary phase and the dark matter problem, which would probably receive an adequate solution within an extended gauge model.

Among the unappealing features of the SM there is certainly the large number of free parameters which should be fixed by experiment, the necessity of several irreducible representations (IRR's) to describe all fermions, and the *ad hoc* introduction of the Higgs field with its quartic potential in order to drive the spontaneous symmetry breaking $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \rightarrow SU(3)_C \otimes U(1)_Q$. Moreover, there is an unexplained triplication of fermion families. The first two of these problems are partially solved by embedding the SM into a unified model corresponding to a larger gauge group, such as $SU(5)$, $SO(10)$ or E_6. However, in doing so, there is an increase in the arbitrariness of the model in two respects: the choice of the unified group, and of a suitable set of Higgs fields to perform all necessary symmetry breaking.

Connes and Lott (CL) have reformulated the SM [1, 2] using the tools of Noncommutative Geometry (NCG) [3], a novel branch of mathematics. Remarkably, in this scheme the Higgs field naturally emerges as a Yang–Mills field, on the same footing of the gauge vector bosons, and the quartic potential, together with the kinetic term, is nothing but its Yang–Mills action [4]-[10]. Furthermore, one gets the correct hypercharge assignments [11], the indication that the number of fermion families must be larger than one, and interesting fuzzy relations among particle masses [10]. The vector behaviour of the strong interactions also emerges naturally [12]. NCG models may also provide for an inflationary phase in the early universe [13].

On the basis of such an interesting result, it is worth investigating the unification programme in the framework of NCG. We have done this analysis under the assumption that the Hilbert space on which the model is constructed is the space of physical fermionic degrees of freedom, namely the observed particles. We did however allow for the presence of right-handed neutrinos. There have been previous attempts in this direction. Notably Chamseddine, Felder and Frölich [14, 15], have succeeded in building unified theories akin to the Connes–Lott model. Unlike CL (and the present paper) however they use an auxiliary Hilbert space which is not the space of physical fermionic degrees of freedom.

We have considered both simple and semisimple groups containing the SM and then applied the CL formalism. In particular, this implies that the fundamental structure to consider is the smaller algebra of matrices containing the chosen group as the set of unitary elements, up to a $U(1)$ factor removed by the unimodularity condition [1, 2]. The IRR's for these algebras, considered as real algebras, are only the fundamental one and its complex conjugate. This rules out all simple groups, like $SU(5)$ or $SO(10)$, for which fermions belong, in general, to non fundamental IRR's. As a further constraint on viable models, in the CL approach the Poincaré duality condition [1, 2] has to be satisfied in order to have gauge invariance. The requirement that both the mentioned conditions are satisfied leads to the conclusion that there are no possible extensions even to larger semisimple groups.

The paper is organized as follows: in section 2 we briefly review the standard model á la Connes–Lott, in the new version of [2]; in section 3 we study all possible extensions of the SM in the framework of NCG. Finally we give our conclusions in section 4.

2 The standard model á la Connes–Lott

We will present here a very brief introduction of the *new version* [2] of the CL model. In the following analysis the general framework introduced in Refs. [10] and [11] will be adopted.

In the usual construction of a gauge theory, several ingredients are required: a space–time manifold M, a gauge symmetry group G, a set of fields defined on M and belonging to some IRR's of G, and a lagrangian density, invariant under the gauge group, ruling

the behaviour of such fields on M. The fields of the model are divided into *matter fields* (fermionic degrees of freedom), *gauge bosons*, and, where spontaneous symmetry breaking is occurring, *Higgs fields* with their quartic potential.

The above construction has a geometric interpretation which allows for a straightforward NCG generalization. In this scheme, the gauge fields define a connection 1-form, the ordinary derivatives are replaced by the covariant ones, and the kinetic term for gauge bosons is given by the square of the curvature associated to the gauge connection.

The programme of NCG [3] is based on the observation that it is possible to study the properties of a manifold M, usually seen as a geometrical set of points, looking at the algebra of complex continuous functions defined on it. This opens the perspective for a noncommutative generalization by considering a noncommutative algebra $\mathcal{A}$. By observing that the topological, metrical, and differential properties of the usual pseudo-riemaniann manifolds are very well captured by the Dirac operator, the basic ingredients of a noncommutative gauge model can be summarized in terms of the real spectral triple $(\mathcal{A}, \mathcal{H}, D)$. $\mathcal{A}$ stands for a $*$-algebra represented on the Hilbert space $\mathcal{H}$, and D (Dirac operator) is an unbounded selfadjoint operator with compact resolvent, such that the commutator of D with any element of $\mathcal{A}$ is a bounded operator. In the case of the standard model the algebra $\mathcal{A}$ is the tensor product of two algebras, $\mathcal{A} = C^\infty(M, \mathbb{C}) \otimes \mathcal{A}_F$ where $C^\infty(M, \mathbb{C})$ is the algebra of smooth functions on M, and $\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$ is the smallest algebra containing the group to be gauged $G = SU(2) \otimes U(1) \otimes U(3)$ as the set of its unitary elements*. With $\mathbb{H}$ we denote the algebra of quaternions, represented as 2×2 matrices

$$\begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \in \mathbb{H} \quad \text{with} \quad x, y \in \mathbb{C} \quad , \tag{2.1}$$

and $M_3(\mathbb{C})$ is the algebra of complex 3×3 matrices. The Hilbert space, $\mathcal{H}$, is the tensor product $\mathcal{H} = L^2(S_M) \otimes \mathcal{H}_F$, and is the space of spinor fields containing both particles and antiparticles. $\mathcal{H}_F$ can be decomposed according to chirality as

$$\mathcal{H}_F = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c \quad , \tag{2.2}$$

*Notice that this group contains, in addition to the SM one, an extra $U(1)$ factor which can be removed by applying the unimodularity condition [1, 2]

where

$$\mathcal{H}_L = \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}\right) \quad , \tag{2.3}$$

$$\mathcal{H}_R = \left((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}\right) \quad , \tag{2.4}$$

and $\mathcal{H}^c_{L,R}$ are the corresponding spaces for antiparticles

$$\mathcal{H}^c_R = \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}\right) \quad , \tag{2.5}$$

$$\mathcal{H}^c_L = \left((\mathbb{C} \oplus \mathbb{C}) \otimes \mathbb{C}^N \otimes \mathbb{C}^3\right) \oplus \left(\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}\right) \quad , \tag{2.6}$$

The Hilbert space $\mathcal{H}_F$ has dimensions equal to the number of different fermions and antifermions, i.e. $30N$ where N is the number of generations. For the moment we do not consider right-handed neutrinos for the SM. A generalization including them and a discussion of massive Dirac neutrinos can be found in [16]. Thus, the natural basis is given by the flavour eigenstate degrees of freedom, namely, for $N = 3$

$$\begin{pmatrix} u_\alpha \\ d_\alpha \end{pmatrix}_L , \begin{pmatrix} c_\alpha \\ s_\alpha \end{pmatrix}_L , \begin{pmatrix} t_\alpha \\ b_\alpha \end{pmatrix}_L , \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L , \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L , \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L , \tag{2.7}$$

$$\begin{matrix} (u_\alpha)_R, & (c_\alpha)_R, & (t_\alpha)_R \\ (d_\alpha)_R, & (s_\alpha)_R, & (b_\alpha)_R, \end{matrix} \; (e)_R,\ (\mu)_R,\ (\tau)_R, \tag{2.8}$$

$$\begin{pmatrix} u^c_\alpha \\ d^c_\alpha \end{pmatrix}_R , \begin{pmatrix} c^c_\alpha \\ s^c_\alpha \end{pmatrix}_R , \begin{pmatrix} t^c_\alpha \\ b^c_\alpha \end{pmatrix}_R , \begin{pmatrix} \nu^c_e \\ e^c \end{pmatrix}_R , \begin{pmatrix} \nu^c_\mu \\ \mu^c \end{pmatrix}_R , \begin{pmatrix} \nu^c_\tau \\ \tau^c \end{pmatrix}_R , \tag{2.9}$$

$$\begin{matrix} (u^c_\alpha)_L, & (c^c_\alpha)_L, & (t^c_\alpha)_L \\ (d^c_\alpha)_L, & (s^c_\alpha)_L, & (b^c_\alpha)_L, \end{matrix} \; (e^c)_L,\ (\mu^c)_L,\ (\tau^c)_L, \tag{2.10}$$

where $\alpha = 1, 2, 3$ is the colour index[†].

A key point in the CL construction is to introduce a real structure on the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. We first consider the two linear isometries ϵ, which has the spaces of particles and antiparticles as eigenspaces with eigenvalues respectively equal to 1 and -1, and the chirality χ. These two operators realize the decomposition of the Hilbert space as in Eq. (2.2) and yield the following properties

$$\epsilon^2 = 1 \; , \quad \chi^2 = 1 \; , \quad \chi = \chi^\dagger \; , \tag{2.11}$$

[†]Note that according to our notations the right-handed antiparticles (2.9) transform as $\overline{\mathbf{2}}$ under $SU(2)_L$, while usually the doublets obtained from (2.9) by applying $i\sigma_2$ are considered, transforming as $\mathbf{2}$.

$$[\epsilon, \chi] = 0\ , \quad [\alpha, \chi] = [\alpha, \epsilon] = 0\ ,\ \forall \alpha \in \mathcal{A}\ , \quad \{\chi, D\} = 0\ , \tag{2.12}$$

A real structure is then an antilinear isometry J of $\mathcal{H}$ satisfying the following properties

$$J^2 = 1\ , \quad \{\epsilon, J\} = [J, D] = 0\ , \quad J\chi = \pm\chi J \quad , \tag{2.13}$$

$$\left[\alpha, J\beta J^\dagger\right] = 0\ , \quad \left[[D, \alpha]\ , J\beta J^\dagger\right] = 0\ , \quad \forall \alpha, \beta \in \mathcal{A} \quad , \tag{2.14}$$

where $\{\ ,\ \}$ denotes anticommutation[‡]. The last two equations (2.14), as explained in [2], are related to Poincaré duality and ensure the gauge invariance of the lagrangian density.

In the following we will restrict our analysis to the finite part of the triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$, where D_F is defined from the Dirac operator D by the relation [11]

$$D = \partial\!\!\!/ \ \otimes \mathbb{I} + \mathbb{I} \otimes D_F \quad . \tag{2.15}$$

Let us now consider an element (a, b, c) of $\mathcal{A}_F$, where a, b and c belong to $\mathbb{H}$, $\mathbb{C}$ and $M_3(\mathbb{C})$ respectively. A faithful representation ρ of $\mathcal{A}_F$ on the Hilbert space $\mathcal{H}_F$ is the following

$$\rho(a, b, c) \equiv \begin{pmatrix} \rho_w(a, b) & 0 \\ 0 & \rho_s^*(b, c) \end{pmatrix} \quad , \tag{2.16}$$

where

$$\rho_w(a, b) \ \equiv \ \begin{pmatrix} a \otimes \mathbb{I}_N \otimes \mathbb{I}_3 & 0 & 0 & 0 \\ 0 & a \otimes \mathbb{I}_N & 0 & 0 \\ 0 & 0 & B \otimes \mathbb{I}_N \otimes \mathbb{I}_3 & 0 \\ 0 & 0 & 0 & b^*\, \mathbb{I}_N \end{pmatrix}, \quad B \equiv \begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix} \tag{2.17}$$

$$\rho_s(b, c) \ \equiv \ \begin{pmatrix} \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c & 0 & 0 & 0 \\ 0 & b^*\, \mathbb{I}_2 \otimes \mathbb{I}_N & 0 & 0 \\ 0 & 0 & \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c & 0 \\ 0 & 0 & 0 & b^*\, \mathbb{I}_N \end{pmatrix}. \tag{2.18}$$

The operators χ_F and J_F have the form

$$\chi_F \ = \ \begin{pmatrix} -\mathbb{I}_{24} & 0 & 0 & 0 \\ 0 & \mathbb{I}_{21} & 0 & 0 \\ 0 & 0 & \mathbb{I}_{24} & 0 \\ 0 & 0 & 0 & -\mathbb{I}_{21} \end{pmatrix}, \tag{2.19}$$

$$J_F \ = \ J_F^\dagger = \begin{pmatrix} 0 & \mathbb{I}_{45} \\ \mathbb{I}_{45} & 0 \end{pmatrix} C \ , \tag{2.20}$$

$$\tag{2.21}$$

[‡]The introduction of such an operator is inspired by Tomita's theorem [17, 18], which, for a Von Neumann algebra with cyclic and separating vector in $\mathcal{H}$, gives an antilinear involution such that $J\mathcal{A}J^\dagger$ is the commutant of the algebra.

where C is the complex conjugation. The role of J_F is to interchange particles with antiparticles and, at same time, chirality. It therefore acts, up to a complex conjugation, as the Dirac charge conjugation $\mathcal{C}$

$$\mathcal{C}\psi_{L,R} = i\gamma_2\psi^*_{L,R} = (\psi^c)_{R,L} \quad , \tag{2.22}$$

where γ_2 is the second Dirac matrix.

The euclidean Dirac operator D_F for the standard model is

$$D_F = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{M}^* \\ 0 & 0 & \mathcal{M}^T & 0 \end{pmatrix} . \tag{2.23}$$

The 24×21 mass matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} M_u \otimes \mathbb{I}_3 & 0 \\ 0 & M_d \otimes \mathbb{I}_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 \\ M_e \end{pmatrix} \end{pmatrix} , \tag{2.24}$$

where

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix} , \quad M_d = C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} , \quad M_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} . \tag{2.25}$$

In the previous relations C_{KM} denotes the Cabibbo–Kobayashi–Maskawa mixing matrix. Notice that the antiparticle mass matrix appearing in the lower right corner of (2.23) is obtained from the corresponding one for particles by using the charge conjugation on bilinear mass terms in the lagrangian density

$$\begin{pmatrix} 0 & \mathcal{M} \\ \mathcal{M}^\dagger & 0 \end{pmatrix} \xrightarrow{\mathcal{C}} \begin{pmatrix} 0 & \mathcal{M}^* \\ \mathcal{M}^T & 0 \end{pmatrix} \quad . \tag{2.26}$$

In general, given a Dirac operator D, the gauge connection is written as

$$A = \sum_i \beta_i [D, \alpha_i] \equiv \sum_i \beta_i d\alpha_i \quad , \tag{2.27}$$

where α_i and β_i are elements of $\mathcal{A}$, and the differential d is defined by $d\alpha \equiv [D, \alpha]$. From the connection A, one defines the curvature θ as[§]

$$\theta \equiv dA + A^2 \quad , \tag{2.28}$$

[§]Note that the d operator so defined is not nilpotent and hence a quotient is necessary in order to to obtain the correct differential algebras [3, 4].

and thus the bosonic lagrangian density is obtained as

$$\mathcal{L}_{\mathcal{B}} = \frac{1}{\mathcal{N}} \mathrm{Tr}\, \theta^2 \quad , \tag{2.29}$$

where $\mathcal{N}$ is a normalization constant.

In order to define the fermionic part of the Lagrangian, let us observe that the operator J enables the definition of a right action of the algebra $\mathcal{A}$ on $\mathcal{H}$ as

$$\Psi\alpha \equiv J\alpha^\dagger J\Psi \quad . \tag{2.30}$$

The first relation in (2.14) ensures that right and left actions commute. We can now define an adjoint action of the gauge group, identified with the set of unitary elements of the algebra, on the Hilbert space

$${}^u\Psi = u\Psi u^\dagger = uJuJ\Psi \quad , \tag{2.31}$$

and the fermionic lagrangian density is

$$\Psi^\dagger(D + A + JAJ)\Psi \quad . \tag{2.32}$$

Using relations (2.14) it is possible to check [11] that (2.32) is invariant under the transformations (2.31) and

$${}^uA = uAu^\dagger + u[D, u^\dagger] \quad . \tag{2.33}$$

For the Dirac operator D defined in Eqs. (2.15), (2.23) and the algebra $C^\infty(M, \mathbb{C}) \otimes \mathcal{A}_F$, where $\mathcal{A}_F$ is represented as in (2.16)–(2.18), one obtains the full Lagrangian of the SM. In particular, the $SU(2)_L$ doublet Higgs field, φ, naturally arises along with its kinetic term and quartic potential

$$V(\varphi) = \frac{K}{16L^2}|\varphi|^4 - \frac{K}{2L}|\varphi|^2 \quad , \tag{2.34}$$

where K and L are known functions of the fermion masses [10]. From the lagrangian density one can obtain some relations among particle masses and coupling constants of SM [10]. In particular, by denoting with g_2 and g_3 the gauge couplings for $SU(2)_L$ and $SU(3)_C$ respectively, one has

$$\sin^2\theta \;<\; \frac{2}{3}\left(1 + \frac{1}{9}\left(\frac{g_2}{g_3}\right)^2\right)^{-1} \quad , \tag{2.35}$$

$$m_e^2 \;<\; m_W^2 < \frac{1}{3}\left(m_t^2 + m_b^2 + m_c^2 + m_s^2 + m_d^2 + m_u^2\right) \approx \frac{1}{3}m_t^2 \quad , \tag{2.36}$$

$$m_H \;=\; (280 \pm 33)\; GeV \quad . \tag{2.37}$$

Note that (2.36) provides a lower bound for the *top* quark mass, and (2.37) has been obtained from a relation which involves the quark masses (*top* included).

3 Unified theories in noncommutative geometry

We have seen that in the CL approach it is possible to obtain a complete description of the SM and, in particular, of the spontaneous symmetry breaking mechanism down to $SU(3)_C \otimes U(1)_Q$ by suitably choosing the structure of the Dirac operator. In this section we will analyze if this approach is compatible with larger gauge group symmetries, which are effective at higher energies, and which at some scale break down to the SM. It is well known that much effort has been devoted to the so called unification programme, namely to find simple gauge groups which contains the SM and are compatible with the low energy phenomenology. The simplest version of the non supersymmetric Georgi and Glashow model, based on the group $SU(5)$, has been ruled out by the accurate experimental results on the strong coupling constant, $\sin^2\theta_W$ at the M_Z scale, and the lower limit on the proton lifetime. Among the groups whose algebra have rank 5, unification based on $SO(10)$ is still consistent with all experimental constraints and also gives interesting predictions for neutrino masses. This latter point is particularly relevant for many topics which are at the border between particle physics and cosmology, like the solar neutrino problem and the nature of dark matter. Many other attempts have been done by considering, for example, exceptional algebras, or larger unitary and orthogonal groups, in which the generation degrees of freedom are gauged[¶]. It is also worth mentioning that in this unification programme, several semisimple groups have been studied as well, which would represent an intermediate step towards a complete unification. Typical examples are the left-right models, as $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R$ or $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$, where $SU(4)_{PS}$ is the Pati-Salam group [20], which for example appears as possible intermediate symmetry stages in the $SO(10)$ breaking to the SM.

In this section we will consider all possible simple and semisimple algebras, which contain the SM, and for which the CL geometrical point of view will be applied. It is remarkable that this approach gives more constraints than the ones which should be

[¶]For a review see [19].

satisfied if one adopts the customary way to construct a spontaneously broken gauge theory. Usually the IRR's which are used to represent fermionic fields are, in general, not constrained to be the fundamental ones of the considered Lie algebras. However, as we have seen, in the CL approach one starts with the smallest algebra of matrices which contains the chosen gauge group as the set of unitary elements. The only IRR's of the gauge group which are allowed are the ones coming from corresponding IRR's of the algebra. It follows that *only the fundamental IRR and its complex conjugate can be used to classify fermions.*

In the analysis we will impose several requirements, some of which are quite obvious and usually assumed. Others, as the one on the dimensionality of IRR's just discussed, are instead more related to the CL construction. In particular:

i) the algebra should contain $\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})$;

ii) we will consider only the usual particle spectrum, the one already present in the SM, with the only possible addition of right-handed neutrinos;

iii) only *fundamental* IRR's can be used to accomodate left and right-handed fermions;

iv) the IRR's should be *complex* in order to allow left and right-handed particles to behave independently under the action of the gauge group;

v) the IRR's should contain only colour triplets and weak isospin doublets of $SU(2)_L$;

vi) all components of the gauge connection, namely gauge vector bosons *and* Higgs fields, should transform as

$$A \to uAu^\dagger + u[D, u^\dagger] \quad . \tag{3.38}$$

Notice that, commonly there are no constraints on the IRR's to be used for Higgs fields but general symmetry requirements on the lagrangian density. The transformation rule (3.38) for the Higgs is peculiar of their geometrical interpretation as a part of the connection on the noncommutative components of the geometrical space. We will stress in the following that (3.38) is consistent with gauge invariance of the interaction terms among fermions and gauge bosons only if the conditions (2.14) are satisfied.

Condition *iv*) can be motivated as follows. In discussing unified theories it is convenient to use as a basis for matter fields fermions and antifermions both left-handed, belonging to one or more IRR's of the gauge group. If we denote this basis with f_L, the right-handed fermions f_R would transform then as f_L^*. Thus, if we had chosen a self-conjugate IRR of the gauge group right and left-handed particles would had transformed in the same way.

Conditions *iii*) and *iv*) rule out the possibility to use orthogonal groups for unified models, since $SO(n)$ fundamental IRR's are all self-conjugate. In particular, it is worth noticing that unification based on $SO(10)$, which is the most appealing non supersymmetric unified model, cannot be realized in the CL approach. For the above reasons we will not consider in the following analysis orthogonal groups.

We point out that all our discussion is still at the classical level. It is well known that appearance of anomalies spoils gauge invariance at quantum level[‖]. This imposes additional constraints on viable unified gauge models. However from our analysis, it appears that already there are no possible choices satisfying all *classical* requirements $i) - vi)$.

We will not consider supersymmetric theories. It would be interesting to extend a similar analysis also to this case. We will start by discussing simple groups and then we will consider the semisimple ones, ordered with increasing rank. In our notations, an IRR for the group $G_1 \otimes ... \otimes G_n$ is denoted by $(\mathbf{d}_1, ..., \mathbf{d}_n)$, where $\mathbf{d}_i$ is the d_i–dimensional IRR of G_i.

3.1 Simple Groups

Rank 4: $SU(5)$

The left-handed fermions are accommodated in two IRR's; $\overline{\mathbf{5}} \oplus \mathbf{10}$. The $\mathbf{10}$ can be obtained as the antisymmetric part in the product $\mathbf{5} \otimes \mathbf{5}$. Choosing the algebra as $M_5(\mathbb{C})$, the $\mathbf{10}$ would not be a IRR of the algebra but only of its group of unitaries. This is a typical case in which Yang–Mills theories have a freedom that Connes–Lott models do not have.

Rank 14 and 15: $SU(15)$ and $SU(16)$

Let us first consider $SU(16)$. For the algebra $M_{16}(\mathbb{C})$ left–handed fermions can form a

[‖]The cancellation of anomalies for the SM has an interesting counterpart in the unimodularity condition in CL [21].

16, while the right–handed ones a $\overline{\mathbf{16}}$, which are complex representations as they should. The problem arises however for the embedding

$$SU(3)_C \otimes SU(13) \otimes U(1) \subset SU(16) \quad . \tag{3.39}$$

In this case the **16** decomposes as

$$\mathbf{16} = (\mathbf{3}, \mathbf{1}, \lambda) \oplus (\mathbf{1}, \mathbf{13}, -\frac{3}{13}\lambda) \quad , \tag{3.40}$$

and this means that coloured states are $SU(2)_L$ singlets, since

$$SU(2)_L \subset SU(13) \quad . \tag{3.41}$$

For $SU(15)$ the discussion is analogous. Instead of (3.39) we have

$$SU(3)_C \otimes SU(12) \otimes U(1) \subset SU(15) \quad , \tag{3.42}$$

and again we find that colour triplets would be $SU(2)_L$ singlets, which is incorrect.

3.2 Semisimple Groups

Rank 5: $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R$

This group was introduced as a relevant example of left–right symmetric model [20]. It also appears as an intermediate stage in the symmetry breaking of $SO(10)$ to the SM. In this model, a fermionic family is accommodated in the following IRR's of the group

$$\begin{array}{ll} \begin{pmatrix} u_\alpha & \nu_e \\ d_\alpha & e \end{pmatrix}_L = (\mathbf{4}, \mathbf{2}, \mathbf{1}) \quad , & \begin{pmatrix} {u^c}_\alpha & \nu^c_e \\ {d^c}_\alpha & e^c \end{pmatrix}_L = (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \quad , \\ \begin{pmatrix} u_\alpha & \nu_e \\ d_\alpha & e \end{pmatrix}_R = (\mathbf{4}, \mathbf{1}, \mathbf{2}) \quad , & \begin{pmatrix} {u^c}_\alpha & \nu^c_e \\ {d^c}_\alpha & e^c \end{pmatrix}_R = (\overline{\mathbf{4}}, \mathbf{2}, \mathbf{1}) \quad . \end{array} \tag{3.43}$$

The model, in its minimal version, requires two Higgs multiplets transforming respectively as $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ and $(\mathbf{1}, \mathbf{2}, \mathbf{2})$. The first one breaks the symmetry from $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R$ to SM, whereas the second multiplet drives the breaking to $SU(3)_C \otimes U(1)_Q$. In general the Yukawa coupling of the Higgs multiplet $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ to right-handed fermions provides Majorana mass terms to right-handed neutrinos.

In order to describe this model in the Connes-Lott framework one chooses the algebra $M_4(\mathbb{C}) \oplus \mathbb{H}_L \oplus \mathbb{H}_R$. This is also consistent with the particle content of the model, since in

the classification of fermion families presented in (3.43), particles belong to fundamental IRR's of the group only. On the Hilbert space

$$\mathcal{H}_F = \bigoplus_{i=1}^{N} \Big((\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\mathbf{4},\mathbf{1},\mathbf{2}) \oplus (\overline{\mathbf{4}},\mathbf{2},\mathbf{1}) \oplus (\overline{\mathbf{4}},\mathbf{1},\mathbf{2}) \Big)_i \tag{3.44}$$

(where i stands for the family index), and denoting with h, a_L and a_R elements of $M_4(\mathbb{C})$, $\mathbb{H}_L$ and $\mathbb{H}_R$ respectively, the algebra $\mathcal{A}_F$ is represented as follows

$$\rho(h, a_L, a_R) \equiv \begin{pmatrix} a_L \otimes \mathbb{I}_4 \otimes \mathbb{I}_N & & & \\ & a_R \otimes \mathbb{I}_4 \otimes \mathbb{I}_N & & \\ & & \mathbb{I}_2 \otimes h^* \otimes \mathbb{I}_N & \\ & & & \mathbb{I}_2 \otimes h^* \otimes \mathbb{I}_N \end{pmatrix} \quad . \tag{3.45}$$

In the first phase transition the Yukawa terms provide a Majorana mass to right-handed neutrinos, while the second phase transition gives Dirac masses to all particles. It is therefore quite natural to write the finite component of the Dirac operator D_F as follows

$$D_F = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^\dagger & 0 & 0 & \mu \\ 0 & 0 & 0 & \mathcal{M}^* \\ 0 & \mu^\dagger & \mathcal{M}^T & 0 \end{pmatrix} \quad , \tag{3.46}$$

where

$$\mathcal{M} = \begin{pmatrix} M_U \otimes \mathbb{I}_4 & 0 \\ 0 & M_D \otimes \mathbb{I}_4 \end{pmatrix} \quad . \tag{3.47}$$

where M_U and M_D are the $N \times N$ mass matrices in the generation space for *Up particles* (u, c, t quarks and neutrinos) and *Down particles* (d, s, b and charged leptons), respectively. As far as the Majorana mass matrix μ, it has entries in the neutrino sector only and can be written as

$$\mu = \begin{pmatrix} 0_{18} & & \\ & \mu_\nu & \\ & & 0_3 \end{pmatrix} \quad , \tag{3.48}$$

where with 0_n we have indicated a $n \times n$ null matrix, and μ_ν is the 3×3 Majorana mass matrix for neutrinos. However, the presence of μ, and in general of elements in D_F connecting the particle sector with the antiparticle one, makes impossible to satisfy the second relation (2.14), as it can be easily checked. This means that the gauge invariance is not guaranteed [2], and one can actually check by an explicit calculation that it is violated. This can be easily seen by studying the way the Higgs bosons corresponding to the choice (3.46) transform under a gauge transformation. Using (2.27) the connection in

the finite component of the complete triple results to be

$$\begin{pmatrix} 0 & \phi & 0 & 0 \\ \phi^\dagger & 0 & 0 & \Phi \\ 0 & 0 & 0 & 0 \\ 0 & \Phi^\dagger & 0 & 0 \end{pmatrix} \quad , \tag{3.49}$$

where we have used the fact that $[M_{U,D} \otimes \mathbb{I}_4, \mathbb{I}_2 \otimes h^* \otimes \mathbb{I}_N] = 0$ According to (3.38) one gets the behaviour of ϕ and Φ under a gauge transformation

$$\begin{aligned} \phi &\rightarrow -\mathcal{M} + a_R^u \, \mathcal{M} \, a_L^{u*} + a_R^u \, \phi \, a_L^{u*} \\ \Phi &\rightarrow -\mu + a_R^u \, \mu \, h^u + a_R^u \, \Phi \, h^u \end{aligned} \tag{3.50}$$

where a_L^u, a_R^u and h^u are unitary elements of $\mathbb{H}_L$, $\mathbb{H}_R$ and $M_4(\mathbb{C})$ respectively. From the previous results, it follows that ϕ transforms as a $(\mathbf{1},\mathbf{2},\mathbf{2})$ IRR under the gauge group, and, as we have mentioned, it is the Higgs needed to break the symmetry from the SM down to $SU(3)_C \otimes U(1)_Q$. The multiplet Φ, instead, transforms as $(\mathbf{4},\mathbf{1},\mathbf{2})$ and not as $(\mathbf{10},\mathbf{1},\mathbf{3})$. Only the latter can be coupled to right-handed fermions and left-handed antifermions in a gauge invariant way via Yukawa terms, while similar terms involving Φ just obtained would spoil gauge invariance of the lagrangian density. It follows that the breaking of the second of conditions (2.14), due to the introduction of μ, i.e. of Majorana mass terms, leads to a model which does not satisfy gauge invariance.

Rank 5: $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$

This case is very similar to the previous one. The algebra is $M_3(\mathbb{C}) \otimes \mathbb{H}_L \otimes \mathbb{H}_R \otimes \mathbb{C}$, and the particle assignments in the various representations are as in the standard model with the only difference that right-handed particles, including neutrinos, are classified in doublets under $SU(2)_R$ and singlet under $SU(2)_L$, the reverse being true for left–handed particles.

Given $a_L \in \mathbb{H}_L$, $a_R \in \mathbb{H}_R$, $c \in M_3(\mathbb{C})$ and $b \in \mathbb{C}$ the representation of the algebra is similar to the one of the standard model

$$\rho(a_L, a_R, b, c) \equiv \begin{pmatrix} \rho_w(a_L, a_R) & 0 \\ 0 & \rho_s^*(b, c) \end{pmatrix} \quad , \tag{3.51}$$

where

$$\rho_w(a_L, a_R) \equiv \begin{pmatrix} a_L \otimes \mathbb{I}_N \otimes \mathbb{I}_3 & 0 & 0 & 0 \\ 0 & a_L \otimes \mathbb{I}_N & 0 & 0 \\ 0 & 0 & a_R \otimes \mathbb{I}_N \otimes \mathbb{I}_3 & 0 \\ 0 & 0 & 0 & a_R \otimes \mathbb{I}_N \end{pmatrix},$$

$$\rho_s(b,c) \quad \equiv \quad \begin{pmatrix} \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c & 0 & 0 & 0 \\ 0 & b^* \, \mathbb{I}_2 \otimes \mathbb{I}_N & 0 & 0 \\ 0 & 0 & \mathbb{I}_2 \otimes \mathbb{I}_N \otimes c & 0 \\ 0 & 0 & 0 & b^* \mathbb{I}_2 \otimes \mathbb{I}_N \end{pmatrix} . \tag{3.52}$$

The Dirac operator D_F has the same form as in (3.46) with μ given by Eq. (3.48) and

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} M_u \otimes \mathbb{I}_3 & 0 \\ 0 & M_d \otimes \mathbb{I}_3 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} M_\nu & 0 \\ 0 & M_e \end{pmatrix} \end{pmatrix} \quad . \tag{3.53}$$

Note that M_u, M_d and M_e are analogously defined as in (2.25), and M_ν is a neutrino Dirac mass matrix (with possible mixing). Again the Majorana mass terms spoil the second of condition (2.14) and so also this model is not viable in the strict CL framework. By reasoning as in the previous case it is easy to show that the two Higgs multiplets ϕ and Φ of Eq. (3.49) transforms under $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R$ as, respectively, $(\mathbf{1},\mathbf{2},\mathbf{2})$ and $(\mathbf{1},\mathbf{1},\mathbf{2})$. The ϕ multiplet has the correct behaviour under the gauge group, while the introduction of Yukawa terms for Φ explicitly breaks the gauge invariance of the lagrangian density. We remind the reader that, in the usual approach, the minimal choice for the Higgs sector of the $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ model involves instead two IRR's transforming as $(\mathbf{1},\mathbf{2},\mathbf{2})$, which we also got in the CL approach, and $(\mathbf{1},\mathbf{1},\mathbf{3})$, which instead does not naturally emerge from it.

Rank 6: $SU(4) \otimes SU(4)$

We start choosing the algebra and the left fermions IRR's as, respectively, $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$ and $(\mathbf{4},\mathbf{4})$, $(\overline{\mathbf{4}},\mathbf{4})$, or $(\mathbf{4},\overline{\mathbf{4}})$. For the embedding

$$SU(3)_C \otimes U(1) \subset SU(4) \quad , \tag{3.54}$$

the 4-dimensional IRR decouples as

$$\mathbf{4} = (\mathbf{1}, 3\lambda) + (\mathbf{3}, -\lambda) \quad . \tag{3.55}$$

If $SU(3)$ is contained in the first $SU(4)$, then $SU(2)$ must be contained in the second one. The two possible embedding corresponding to maximal subalgebras are

i) $SU(2)_L \otimes SU(2) \subset SU(4)$. In this case $\mathbf{4}$=$(\mathbf{2},\mathbf{2})$, and this would mean that the fermions would be doublets also under the second $SU(2)$ factor.

ii) $SU(2)_L \otimes SU(2) \otimes U(1) \subset SU(4)$. In this case $\mathbf{4}$=($\mathbf{2}$,$\mathbf{1}$,λ) +($\mathbf{1}$,$\mathbf{2}$,$-\lambda$).

In both cases only quarks or antiquarks can be accommodated, respectively, in the $\mathbf{4}$ or $\overline{\mathbf{4}}$ IRR's, since only the $\mathbf{3}$ or $\overline{\mathbf{3}}$ of $SU(3)_C$ is present. The 16 particle states, both left and right-handed, would be accommodated into the ($\mathbf{4}$,$\mathbf{4}$), while the antiparticles in the ($\overline{\mathbf{4}}$, $\overline{\mathbf{4}}$). Left and right fermions would therefore appear in the same IRR and do not transform independently, but would mix under a gauge transformation.

Rank 8: $SU(8) \otimes SU(2)$

The algebra in this case is $M_8(\mathbb{C}) \oplus \mathbb{H}$. The IRR's can be accordingly chosen as ($\mathbf{8}$,$\mathbf{2}$) or ($\overline{\mathbf{8}}$,$\mathbf{2}$), which contain all left or right-handed fermions, or ($\mathbf{8}$,$\mathbf{1}$) and ($\overline{\mathbf{8}}$,$\mathbf{1}$). We cannot choose the $SU(2)_L$ of the SM to be the $SU(2)$ factor, because in this case fermions would be all doublets or singlets. Therefore both factors $SU(3)_C$ and $SU(2)_L$ should be contained in $SU(8)$. For the embedding

$$SU(3)_C \oplus SU(5) \oplus U(1) \subset SU(8) \quad , \tag{3.56}$$

the $\mathbf{8}$ decomposes as

$$(\mathbf{3}, \mathbf{1}, 5\lambda) \oplus (\mathbf{1}, 5, -3\lambda) \quad . \tag{3.57}$$

Since $SU(2)_L \subset SU(5)$ all coloured states would be weak isospin singlets.

4 Conclusions

In this paper we have analyzed the possibility to construct a Grand Unified Theory in the strict framework of the new version of the Connes-Lott model. We have assumed the minimal Hilbert space, made of the degrees of freedom of the fermionic particles already observed, allowing only for the existence of right-handed neutrinos. Since the CL model requires the fermions to be in the fundamental IRR's of the gauge group, this selects only two groups as possible unified models beyond the electroweak standard model. In particular, they turn out to be $SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R$ and $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$. Remarkably, these groups appear as the left-right symmetric intermediate stages of $SO(10)$, although $SO(10)$ itself cannot be realized as a CL model. However, also these models have troubles since the Dirac operator causing the correct

spontaneous symmetry breaking to the standard model, must contain Majorana mass terms which spoil gauge invariance of the theory.

On one side, it is an appealing feature of this theoretical framework, the Connes-Lott approach, to be able to select among the possible gauge groups. On the other hand the fact that only the Standard Model has survived our analysis is at variance with the currently accepted idea that new physics beyond the standard model is required at energy scales lower than Planck mass. Thus this analysis seems to suggest that a modification is needed either in the basic ingredients of the model, or in the Hilbert space, which could contain some extra particles, living at higher energy scales.

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